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Surplus Management under a Stochastic Process:

Asset Allocation within a State-Security Approach

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ABSTRACT

This paper proposes a profile of state contingent claims, embedded in a stochastic interest rate process, for the surplus management of an insurance company as an optimal asset allocation strategy. Proper positions of securities based on interest rate situations can be arranged by a surplus manager to fulfill the liability schedule under the pre-specified solvency ability. By considering each path immunization, this asset allocation modeling could be carried into the “multi-period scenarios-based programming model”. Hence, we develop the strategy to implement the concept of path-immunization for the insurance company. Furthermore, we illustrate the impact of the change of the market current term/volatility structure of asset/liability return on the surplus value, a way how to reallocate assets and a hedging strategy for this insurance company in the market with all the state contingent claims needed.

Keywords: Surplus management, Asset allocation, State contingent claim

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I. INTRODUCTION

Fluctuations of the surplus value, equal to the asset value minus the liability value, have been studied since 1952. Considering especially with the interest rate risk, Redington (1952) linked the surplus management with the immunization strategy via Taylor’s expansion. He demonstrated that ignoring higher order terms, first-order zero sensitivity with respect to the market current interest rate level and second-order positive one cause positive changes of the surplus value while the market current interest rate level deviates. The concept of immunization here is that: the surplus value increases regardless of the courses of interest rate movements (Fisher and Weil, 1971; Bierwag, 1977; Bierwag and Kaufamn, 1977). Other related researches further recognized and introduced the specified stochastic interest rate process to investigate the immunization strategy of an insurance company. Boyle (1978) introduced the bond portfolio immunization under a stochastic interest rate process while early studies focused only on a single bond immunization with a deterministic interest rate function. Tzeng, Wang and Soo (2000) adopted an optimization framework to seek a profile of multi-period immunization strategy which fulfills the liability schedule under the pre-specified solvency ability. However, the stochastic interest rate process mentioned above so far, in effect, determines the proper discount factors for different time periods. They did not recognize the fact that the interest rate with different values at different times is truly a dynamic process. Chiu and Lee (2007), filling up this gap, allocated the different amount of asset on the different interest rate path, so called the “multi-period scenarios-based strategy”, to capture a truly dynamic property of interest rate process. Along with this line of research, this paper proposes alternative approach to capture this important property via a profile of state contingent claims. Hence, the derivatives-based hedging strategy of an insurance company is suggested, even when the economic environment changes (e.g., the market current term/volatility structure of asset/liability return changes).

In general, driving forces of uncertainty are described by stochastic interest rate models. One paradigm of stochastic interest rate models is that of no arbitrage model (Ho and Lee, 1986; Black, Derman, and Toy, 1990; Black and Karasinski, 1991; Hull and White, 1990) which utilizes the full information of the market current term/volatility structure, and the other one is that of equilibrium model

(Vasicek, 1977; Cox, Ingersoll, and Ross, 1985) which usually requires the estimation of the market price of risk, and is not well adapted to the market current term/volatility structure. The others deal with different underlying markets, such as the forward rate market (Heath, Jarrow, and Morton, 1992) and the swap rate market (Brace, Gatarek, and Musiela, 1997). Due to the importance of the market current term/volatility structure, this paper focuses only on that of no arbitrage model. Formally, this paper considers a profile of state contingent claims, embedded in a no-arbitrage stochastic interest rate process, for the surplus management of an insurance company to fulfill the liability schedule under the pre-specified solvency ability.

For the purpose of easy exposition, this paper introduces the stochastic interest rate process suggested by Black, Derman, and Toy (1990) to examine the immunization strategy for an insurance company. We calculate the company’s surplus value and the first/second order sensitivity of the surplus value, and also show that a profile of state contingent claims in this paper straightly decomposes a profile of straight bonds, which was the immunization strategy suggested by Tzeng, Wang, and Soo (2000). Further, if a firm’s objective is to maximize its convexity of the surplus value subject to non-anticipated strategy condition, solvency ability, its first order zero sensitivity, and its budget constraint, this paper demonstrates that this optimal immunization strategy within a profile of state contingent claims can be implemented by the multi-period scenarios-based programming model (Chiu and Lee, 2007). Moreover, we show that the cost/benefit of hedging strategy via an optimal immunization strategy only reflects the value change for the economic environmental movements in the market with all the state contingent claims needed.

The paper is organized as follows. Section 2 describes the economic settings of the environment and the model. Section 3 provides a numerical example of a hypothetical insurance company to demonstrate its corresponding hedging strategy. Section 4 concludes.

II. THE MODEL

Assume that the market current term/volatility structure is well observed both on the asset return and the liability return, and these settings can be described by the binomial interest rate tree model suggested by Black, Derman, and Toy (1990), so called the 'binomial BDT tree model'. In practice, for example, the market current term structure of asset return is estimated by the term structure of treasury bond market plus the proper term structure of credit spread for its corresponding asset portfolio in the corporate bond market. The volatility structure of asset return could also be estimated from the market asset returns. Similarly, it is in the same way for those of liability return, except that with no explicit market information, hence it further requires some kinds of subjective judgments. Accordingly, assume that the market current term structure of asset/liability return can be expressed as follows:

$$Y_A(\tau) = E_A - F_A \exp(-G_A \cdot \tau) \tag{1}$$

where Y_A is the continuously compounded yield for the asset return, τ is the time to maturity, and E_A, F_A, and G_A are constants, and,

$$Y_L(\tau) = E_L - F_L \exp(-G_L \cdot \tau) \tag{2}$$

where Y_L is the continuously compounded yield for the liability return, τ is the time to maturity, and E_L, F_L, and G_L are constants. The parameter G represents a growth rate, the parameter F represents a scaling factor to control the initial slope of the market current term structure, and the parameter E represents the parallel shift of the market term structure of asset/liability return, which is consistent with the market current asset/liability return level. On the binomial BDT tree, the short rate volatility of ln r(t)[1] (for the purpose of easy exposition, it is not yield volatility as suggested in original paper of Black, Derman, and Toy (1990)), σ, depends on the time interval Δt and is equal to $(1/[2*\mathrm{sqrt}(\Delta t)]) \cdot \ln(r^U / r^D)$, where r^U is the upward interest rate and r^D is the downward interest rate. Assume that the volatility structure of asset/liability return can be expressed as

$$[\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_N]_A \tag{3}$$

where σ_i is the spot *i*th period volatility for the asset return, and,

$$[\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_N]_L \tag{4}$$

where σ_i is the spot *i*th period volatility for the liability return. By these

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[1] In the paper of Jamshidian (1991), the limit version of BDT can express as $r(t) = \mu(t) \exp(\sigma(t)W(t))$, where r(t) is a short rate, $\mu(t)$ and $\sigma(t)$ are all deterministic processes, and W(t) is a standard Wiener process.

assumptions mentioned above, this paper calibrates the binomial BDT tree both on the asset and the liability side. Thus, the asset and the liability continuously compounded period returns have been fully constructed.

With considering the surplus immunization, especially with respect to the market current interest rate level r, we assume that there is a linear relationship between the market current asset/liability return level (r_A, r_L) and the market current interest rate level r (e.q. $r_A = I_A + C_A \cdot r$; $r_L = I_L + C_L \cdot r$). That is: the change in the market current interest rate level have different effect on the spot rate of the asset return and the spot rate of the liability return, respectively; i.e. $dr_A / dr = C_A$ and $dr_L / dr = C_L$, where r_A is the market current asset continuously compounded return level, r_L is the market current liability continuously compounded return level, C_A and C_L are constants (the same settings as Tzeng, Wang, and Soo, 2000).

The followings demonstrate the reasonable state contingent claims embedded in the binomial BDT tree. For the purpose of easy exposition, we take a four-period binomial BDT tree of asset return with one year per period, as an example, to show the reasonable number of state contingent claims. There are 16 kinds of paths in this BDT tree, denoted by the notations of $\omega_1, \omega_2, \dots, \omega_{16}$, where each "scenario *i*" ω_i represents the *i*th path of BDT tree. And, this BDT tree generates 47 state contingent claims, that is 1+2+4+8+16+16 = 47 state contingent claims[2]. In general, we can provide the general form of the number of state contingent claims with respect to the number of period *n* in a *n*-period binomial interest rate model. The general form is $\sum_{k=0}^{n} 2^k + 2^n$. For each kind of state contingent claim, we define its corresponding state of BDT tree as follows:

Time 0, $\{[\omega_1, \omega_2, \dots, \omega_{16}]$(state 1)$\}$;

Time 1, $\{[\omega_1, \omega_2, \dots, \omega_8]$(state 2), $[\omega_9, \omega_{10}, \dots, \omega_{16}]$(state 3)$\}$;

Time 2, $\{[\omega_1, \omega_2, \dots, \omega_4]$(state 4), $[\omega_5, \omega_6, \dots, \omega_8]$(state 5), $[\omega_9, \omega_{10}, \dots, \omega_{12}]$(state 6), $[\omega_{13}, \omega_{14}, \dots, \omega_{16}]$(state 7)$\}$;

Time 3, $\{[\omega_1, \omega_2]$(state 8), $[\omega_3, \omega_4]$(state 9), $[\omega_5, \omega_6]$(state 10), $[\omega_7, \omega_8]$(state 11), $[\omega_9, \omega_{10}]$(state 12), $[\omega_{11}, \omega_{12}]$(state 13), $[\omega_{13}, \omega_{14}]$(state 14), $[\omega_{15}, \omega_{16}]$(state 15)$\}$;

Time 4, $\{[\omega_1]$(state 16), $[\omega_2]$(state 17), $[\omega_3]$(state 18),

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2 We show a visual relationship between path scenarios and their corresponding state contingent claims of BDT tree model in Appendix.

$[\omega_4]$(state 19), $[\omega_5]$(state 20), $[\omega_6]$(state 21),
$[\omega_7]$(state 22), $[\omega_8]$(state 23), $[\omega_9]$(state 24),
$[\omega_{10}]$(state 25), $[\omega_{11}]$(state 26), $[\omega_{12}]$(state 27),
$[\omega_{13}]$(state 28), $[\omega_{14}]$(state 29), $[\omega_{15}]$(state 30), $[\omega_{16}]$(state 31)};

Time 5, {$[\omega_1]$(state 32), $[\omega_2]$(state 33), $[\omega_3]$(state 34),
$[\omega_4]$(state 35), $[\omega_5]$(state 36), $[\omega_6]$(state 37),
$[\omega_7]$(state 38), $[\omega_8]$(state 39), $[\omega_9]$(state 40),
$[\omega_{10}]$(state 41), $[\omega_{11}]$(state 42), $[\omega_{12}]$(state 43),
$[\omega_{13}]$(state 44), $[\omega_{14}]$(state 45), $[\omega_{15}]$(state 46), $[\omega_{16}]$(state 47)}; (5)

Take "time 0, state 1" for example, due to the initial point of the binomial BDT tree, all paths pass through state 1. Path 9 -- 16 pass through state 3 which represents the interest rate going up one year later, meanwhile path 1 -- 8 pass through state 2 which represents the interest rate going down one year later. Hence, in this way, each state represents one kind of path for the interest rate going (e.g. up-down-up path, state 13).

Furthermore, each state, say state *i*, could have its corresponding state contingent claim, say the state *i* contingent claim, whose state price is denoted by SP_i. According to the modern financial theory, each state contingent claim, say the state 13 contingent claim, could be evaluated by

$$E^Q(\,PV(\omega)\,1_{\{state\ 13\}}(\omega)\,) = \sum PV(\omega)\cdot p(\omega),\ \omega\in\{state\ 13\} \tag{6}$$

where Q is the risk neutral probability measure, $PV(\omega)$ is the discount factor along with the path ω, $1_{\{.\}}$ is the indicator function, and $p(\omega)$ is the occurance probability of the path ω. Therefore, there are 47 kinds of state contingent claims embedded in this binomial BDT tree. And, we cosider the surplus management of an insurance company in the market with all the state contingent claims needed, (i.e. one could long/short a profile of state contingent claims).

The surplus value, E, is set equal to the asset value, A, minus the liability value, L. Assume that

$$A = \sum A(i)\,SP_i,\ \text{for all state } i, \tag{7}$$

where A(i) is the asset amount investing in the state *i* contingent claim, and SP_i is the value of this kind of security,

$$L = \sum L(i)\,P(i),\ \text{for all period } i, \tag{8}$$

where L(i) is the liability amount investing in the period *i* zero bond, and P(i) is the value of this bond. These settings are the same as Tzeng, Wang, and Soo (2000) except that we decompose the zero straight bonds on the asset side into several state contingent claims. Hence, under the liability schedule L(0), L(1),…, L(5), the corresponding surplus value function would be

$$E = \sum A(i)\, SP_i - \sum L(i)\, P(i). \tag{9}$$

Within the framework of Tzeng, Wang, and Soo (2000), the objective function for immunization is to maximize,

$$\sum A(i)\partial^2 SP_i / \partial r^2 - \sum L(i)\partial^2 P(i) / \partial r^2, \tag{10}$$

subject to

1.) the budget constraint of the asset value,

$$\sum A(i) SP_i = E + L, \tag{11}$$

2.) the first oder zero sensitivity with respect to the market current interest rate level,

$$\sum A(i)\partial SP_i / \partial r = \sum L(i)\partial P(i) / \partial r, \tag{12}$$

3.) the second order positive sensitivity with respect to the market current interest rate level,

$$\sum A(i)\partial^2 SP_i / \partial r^2 \geqq \sum L(i)\partial^2 P(i) / \partial r^2, \tag{13}$$

4.) the solvency ability,

Cash amount is larger than the minimum solvency margin K as time goes by.

5.) the non-negative strategy,

$$A(i) \geqq 0, \text{ for all state } i, \tag{14}$$

This solvency ability constraint is tedious but straight, take “time3, state 13” of the four-period binomial BDT tree as an example. This state is corresponding to the up-down-up path (i.e. r_0-r_1^u-r_2^d-r_3^u), and, along with this path, net cash flow at time 0 is A(1)-L(0), at time 1 is A(3)-L(1), at time 2 is A(6)-L(2), and at time 3 is A(13)-L(3). And, all net cash flows are carried on into the time-state of “time 3, state 13” , and are required to be larger than the minimum solvency margin K. By this path, its constraint would be

$$([(A(1)-L(0)) \cdot (1+r_0)+A(3)-L(1)] \cdot (1+r_1^u)+A(6)-L(2)) \cdot (1+r_2^d)+A(13)-L(3) \geqq K \tag{15}$$

However, we need some kind of operatonal definition of these constraints within this optimization framework. Fortunately, the local property of interest rate shows that $[(1+r_0), (1+r_1^u), (1+ r_2^d)]$ is equal to $[1/2 \cdot SP_1/SP_3, 1/2 \cdot SP_3/SP_6, 1/2 \cdot SP_6/SP_{13}]$. Hence, equation (15) can be rewritten as follows:

$$1/2(1/2[1/2(A(1)\text{-}L(0))SP_1/SP_3+A(3)\text{-}L(1)]SP_3/SP_6+A(6)\text{-}L(2))SP_6/SP_{13}+A(13)\text{-}L(3)\geqq K, \quad (16)$$

In this way, one could write down all constraints of the solvency ability and complete the whole settings of the optimization framework. Especially, equations (10)--(13) and (16) are all linear functions with respect to A(i). Hence, the linear programming can slove this problem.

In effect, one can define the state contingent claims as the straight zero bonds, then equations (7)--(14) would be the same settings as those of Tzeng, Wang, and Soo (2000). However, in this case, these settings are not for a complete path-immunization strategy. To solve this problem, this paper decomposes these straight zero bonds into several state contingent claims and hence immunizes by path for the surplus value of the insurance company. According the four-period binomial BDT tree through the pricing formula, e.g. equation (6), one can rewrite equation (7) as follows:

$$\sum_{j=0}^{5}\sum_{i=1}^{16} p(\omega_i)\, A_j(\omega_i)\, PV_j(\omega_i), \quad (17)$$

subject to so called the "non-anticipating strategy" constraint. The "non-anticipating strategy" constraint is that: the strategy amount is the same as in the same information set, which is described by equation (5) (see also, Chiu and Lee, 2007). In this case, we write the "non-anticipating strategy" constraint as follows:

$$\begin{aligned}
&A_0(\omega_1) = \ldots = A_0(\omega_{16});\\
&A_1(\omega_1) = \ldots = A_1(\omega_8);\ A_1(\omega_9) = \ldots = A_1(\omega_{16});\\
&A_2(\omega_1) = \ldots = A_2(\omega_4);\ A_2(\omega_5) = \ldots = A_2(\omega_8);\\
&A_2(\omega_9) = \ldots = A_2(\omega_{12});\ A_2(\omega_{13}) = \ldots = A_2(\omega_{16});\\
&A_3(\omega_1) = A_3(\omega_2);\ A_3(\omega_3) = A_3(\omega_4);\ A_3(\omega_5) = A_3(\omega_6);\ A_3(\omega_7) = A_3(\omega_8);\\
&A_3(\omega_9) = A_3(\omega_{10});\ A_3(\omega_{11}) = A_3(\omega_{12});\ A_3(\omega_{13}) = A_3(\omega_{14});\ A_3(\omega_{15}) = A_3(\omega_{16}), \quad (18)
\end{aligned}$$

where $A_j(\omega_i)$ is the asset amount allocated on path ω_i at the *j*th period, and $PV_j(\omega_i)$ is the discount factor on path ω_i at the *j*th period. Hence, under the liability schedule L(0), L(1), …, L(5), surplus value function can be rewritten as follows:

$$E = \sum_{i=1}^{16}\sum_{j=0}^{5} p(\omega_i)\ A_j(\omega_i)\ PV_j(\omega_i) - \sum_{j=0}^{5} L(j)\ P(j). \tag{19}$$

Thus, with the “non-anticipated strategy” constraint, equation (10) will be

$$\sum_{i=1}^{16}\sum_{j=0}^{5} p(\omega_i)\ A_j(\omega_i)\partial^2\ PV_j(\omega_i)/\partial r^2 - \sum_{j=0}^{5} L(j)\partial^2\ P(j)/\partial r^2. \tag{20}$$

Similarly, equations (11) -- (13) are

$$\sum_{i=1}^{16}\sum_{j=0}^{5} p(\omega_i)\ A_j(\omega_i)\ PV_j(\omega_i) = E+L, \tag{21}$$

and

$$\sum_{i=1}^{16}\sum_{j=0}^{5} p(\omega_i)\ A_j(\omega_i)\partial PV_j(\omega_i)/\partial r = \sum_{j=0}^{5} L(j)\partial P(j)/\partial r. \tag{22}$$

and

$$\sum_{i=1}^{16}\sum_{j=0}^{5} p(\omega_i)\ A_j(\omega_i)\partial^2\ PV_j(\omega_i)/\partial r^2 \geqq \sum_{j=0}^{5} L(j)\partial^2\ P(j)/\partial r^2. \tag{23}$$

If the path-by-path immunization is matter, and the left hand side of equations (21) -- (23) is on asset-path separately, one could consider the path-version constraints as follows:

$$\sum_{j=0}^{5} A_j(\omega_i)\ PV_j(\omega_i) = E+L,\ \text{for all}\ \omega_i, \tag{24}$$

and

$$\sum_{j=0}^{5} A_j(\omega_i)\partial PV_j(\omega_i)/\partial r = \sum_{j=0}^{5} L(j)\partial P(j)/\partial r,\ \text{for all}\ \omega_i, \tag{25}$$

and

$$\sum_{j=0}^{5} A_j(\omega_i)\partial^2 PV_j(\omega_i)/\partial r^2 \geqq \sum_{j=0}^{5} L(j)\partial^2 P(j)/\partial r^2,\ \text{for all}\ \omega_i, \tag{26}$$

Finally, consider the solvency constraint. Again, take the interest rate up-down-up path (i.e. r_0-r_1^u-r_2^d-r_3^u) as an example. Along with this path, say ω, $PV_0(\omega)$ is 1; $PV_1(\omega)$ is $1/(1+r_0)$; $PV_2(\omega)$ is $1/(1+r_0)\cdot 1/(1+r_1^u)$; $PV_3(\omega)$ is $1/(1+r_0)\cdot 1/(1+r_1^u)\cdot 1/(1+r_2^d)$. Accordingly, $[PV_0(\omega)/PV_3(\omega), PV_1(\omega)/PV_3(\omega), PV_2(\omega)/PV_3(\omega)]$ is equal to $[(1+ r_0)\cdot(1+ r_1^u)\cdot(1+ r_2^d), (1+ r_1^u)\cdot(1+ r_2^d), (1+ r_2^d)]$. One could rewrite equation (14) as follows:

$$(A_0(\omega)-L(0))PV_0(\omega)/PV_3(\omega)+(A_1(\omega)-L(1))PV_1(\omega)/PV_3(\omega)+(A_2(\omega)-L(2))PV_2(\omega)/PV_3(\omega)+(A_3(\omega)-L(3) \geqq K \tag{27}$$

Therefore, these settings, equations (18), (20), (24)--(27) plus non-negative $A_j(\omega_i)$ strategy, are the same settings as the “multi-period scenarios-based programming model” suggested by Chiu and Lee (2007). Hence, this paper actually provides the solid economic meaning of the “multi-period scenarios-based programming strategy”, which in essence is a profile of state contigent claims.

Next section, we will take a numerical example to implement an optimal immunization strategy via the “multi-period scenarios-based programming model”, but explain the corresponding hedging strategy from this new point of view － a profile of state contingent claims, not by the path-by-path scenario point of view as did by Chiu and Lee (2007). Furthermore, we add the impact of volatility structure on the surplus value while Chiu and Lee (2007) did not illustrate these results.

III. NUMERICAL EXAMPLE AND HEDGING STRATEGY

The previous section illustrates the model of asset allocation within a state contingent claim approach for the surplus management of an insurance company. To implement this strategy, we construct a hypothetical insurance company. The balance sheet for a hypothetical insurance company at current time is constructed as shown in Table 1. Without loss of generality, we assume the liabily schedule of a hypothetical insurance company as shown in Table 2 (see also, Tzeng, Wang, and Soo, 2000).

Considering surplus immunization especially with respcet to the market current interest rate level r, we assume that there is a linear relationship between the market current asset/liability return level (r_A, r_L) and the market current interest rate level r (e.q. $r_A = I_A + C_A \cdot r$; $r_L = I_L + C_L \cdot r$). That is: the change in the market current interest rate level have different effect on the spot rate of the asset return and the spot rate of the liability return, respectively; i.e. $dr_A / dr = C_A$ and $dr_L / dr = C_L$, where r_A is the market current asset continuously componded return level, r_L is the market current liability continuously componded return level, C_A and C_L are constants (the same settings as Tzeng, Wang, and Soo, 2000). We assume that the market current continuously compounded interest rate level r is equal to 5.1 %.

Table 1　Balance Sheet of a Hypothetical Insurance Company

Asset	Liability	Surplus
3,382,681	2,882,681	500,000

Table 2 Liability Schedule of the Hypothetical Insurance Company

Periods	Liabilities
1	591,500
2	633,700
3	677,400
4	723,500
5	775,800

Table 3 Economic Parameter Settings

The market current interest rate level r 5.1%		
The market current asset/liability return level (given)	r_A 6.20%	r_L 5.1%
The intercept term of linear relationship	I_A 0.01	I_L 0
The slope term of linear relationship	C_A 1.02	C_L 1
The parallel shift factor (given)	E_A 7.06%	E_L 5.96%
The scaling factor	F_A 0.01	F_L 0.01
The growth rate	G_A 0.15	G_L 0.15

Also assume that $r_A = 0.01 + 1.02 \cdot r$ and $r_L = r$; i.e., I_A=0.01, I_L=0, C_A=1.02, and C_L=1. Further assume that E_A=7.06 %, F_A=0.01, G_A=0.15, E_L=5.96 %, F_L=0.01, G_L=0.15, $[\sigma]_A$=[0.05, 0.05, 0.05, 0.05], and $[\sigma]_L$=[0.05, 0.05, 0.05, 0.05], as shown in Table 3. Finally, the minimum solvency margin K is assumed to be 100,000.

Here, we formally restate the immunization framework in our model settings as follows (see also, Chiu and Lee, 2007):

$$\text{Max} \quad \sum_{i=1}^{16}\sum_{j=0}^{5} p(\omega_i)A_j(\omega_i)\partial^2 PV_j(\omega_i)/\partial r^2 \quad [-\sum_{j=0}^{5} L(j)\partial^2 P(j)/\partial r^2 \ ].$$

Subject to

1.) the "non-anticipating strategy" constraint,

$$A_0(\omega_1) = \dots = A_0(\omega_{16});$$
$$A_1(\omega_1) = \dots = A_1(\omega_8);\ A_1(\omega_9) = \dots = A_1(\omega_{16});$$
$$A_2(\omega_1) = \dots = A_2(\omega_4);\ A_2(\omega_5) = \dots = A_2(\omega_8);$$
$$A_2(\omega_9) = \dots = A_2(\omega_{12});\ A_2(\omega_{13}) = \dots = A_2(\omega_{16});$$
$$A_3(\omega_1) = A_3(\omega_2);\ A_3(\omega_3) = A_3(\omega_4);\ A_3(\omega_5) = A_3(\omega_6);\ A_3(\omega_7) = A_3(\omega_8);$$
$$A_3(\omega_9) = A_3(\omega_{10});\ A_3(\omega_{11}) = A_3(\omega_{12});\ A_3(\omega_{13}) = A_3(\omega_{14});\ A_3(\omega_{15}) = A_3(\omega_{16}),$$

2.) the budget constraint of the asset value,

$$\sum_{j=0}^{5} A_j(\omega_i)\, PV_j(\omega_i) = E+L, \text{ for all } \omega_i,$$

3.) the first oder zero sensitivity with respect to the market current interest rate level,

$$\sum_{j=0}^{5} A_j(\omega_i)\,\partial PV_j(\omega_i) / \partial r = \sum_{j=0}^{5} L(j)\,\partial P(j) / \partial r, \text{ for all } \omega_i,$$

4.) the second order positive sensitivity with respect to the market current interest rate level,

$$\sum_{j=0}^{5} A_j(\omega_i)\,\partial^2 PV_j(\omega_i) / \partial r^2 \geqq \sum_{j=0}^{5} L(j)\,\partial^2 P(j) / \partial r^2, \text{ for all } \omega_i,$$

5.) the solvency ability,

$$\sum_{j=0}^{k} (A_j(\omega_i) - L(j))PV_j(\omega_i)/PV_k(\omega_i) \geqq K, \quad \text{for all} \quad i=1,2,\ldots,16,$$

k=1,2,3,4,5

6.) the non-negative strategy

$$A_j(\omega_i) \geqq 0, \text{ for all } \omega_i, \tag{28}$$

Equation (28) can be solved by the linear programming technique. The results of the linear programming could be expressed as a profile of state contingent claims, as shown in Table 4. That is, the hypothetical insurance company should long the state contingent claims with stated positions. Then, this company could fulfill the liability schedule (see Table 2) under pre-specified solvency ability (the minimum solvency margin assumed to be 100,000). Most importantly, regardless the courses of interest rate level movements, the surplus value of the hypothetical insurance company always increases along with each path of asset return, as shown in Table 5.

Table 4 Optimal Asset Allocation (a profile of state contingent claims)

0		1		2		3		4		5	
state price	amount	state price	amount	state price	amount	state price	amount	state price	amount	state price	amount
SP_1 1.000	A(1) **1236548**	SP_2 0.470	A(2) 0	SP_4 0.221	A(4) 0	0.104	542713	0.049	722898	0.046	1518658
								0.049	852166	0.046	1389867
						0.104	845825	0.049	399552	0.046	1541251
								0.049	529236	0.046	1412152
				SP_5 0.221	A(5) 500835	0.103	6722	0.048	722835	0.046	1560207
								0.048	854114	0.045	1429521
						0.103	311741	0.048	395312	0.045	1585608
								0.048	527046	0.045	1454602
		SP_3 0.470	A(3) **342999**	SP_6 0.220	A(6) **0**	0.103	144606	0.048	722835	0.045	1584463
								0.048	856155	0.045	1451745
						0.103	**453360**	0.048	**391302**	0.045	**1610179**
								0.048	525078	0.044	1477143
				SP_7 0.220	A(7) 305811	0.102	0	0.048	520952	0.044	1631566
								0.048	656504	0.044	1496763
						0.102	294149	0.047	201360	0.044	1660524
								0.047	337410	0.043	1525394

A(u) is the amount of state u contingent claim, SP_u is state price of state u contingent claim

Table 5 Path immunization effect for a profile of state contingent claims

Secnario(ω)	-70 bps	-50 bps	-20 bps	20 bps	50 bps	70 bps
1	0.0174554%	0.0088841%	0.0014164%	0.0014105%	0.0087748%	0.0171549%
2	0.0182479%	0.0092848%	0.0014797%	0.0014718%	0.0091570%	0.0178976%
3	0.0212950%	0.0108271%	0.0017235%	0.0017119%	0.0106392%	0.0207797%
4	0.0222518%	0.0113107%	0.0017998%	0.0017859%	0.0110992%	0.0216733%
5	0.0278877%	0.0141680%	0.0022527%	0.0022339%	0.0138689%	0.0270673%
6	0.0288510%	0.0146548%	0.0023294%	0.0023083%	0.0143315%	0.0279658%
7	0.0325398%	0.0165208%	0.0026242%	0.0025981%	0.0161194%	0.0314401%
8	0.0337012%	0.0171072%	0.0027166%	0.0026876%	0.0166753%	0.0325191%
9	0.0357956%	0.0181780%	0.0028884%	0.0028619%	0.0177580%	0.0346434%
10	0.0367684%	0.0186694%	0.0029659%	0.0029370%	0.0182247%	0.0355496%
11	0.0404852%	0.0205492%	0.0032627%	0.0032287%	0.0200240%	0.0390455%
12	0.0416580%	0.0211413%	0.0033560%	0.0033191%	0.0205847%	0.0401337%
13	0.0464201%	0.0235537%	0.0037378%	0.0036965%	0.0229136%	0.0446653%
14	0.0476013%	0.0241499%	0.0038317%	0.0037874%	0.0234777%	0.0457598%
15	0.0519245%	0.0263349%	0.0041764%	0.0041256%	0.0255619%	0.0498066%
16	0.0533462%	0.0270521%	0.0042892%	0.0042348%	0.0262385%	0.0511184%

The number in the first row represents some basis points deviation from current interest rate level. And, the value in the table represents the rate of change of surplus value for the hypothetical insurance company.

Table 6 Hedging Strategy While the Market Current Term Structure Changes

0	1	2	3	4	5	0	1	2	3	4	5
Parallel shift up (40 bps)						**Parallel shift down (40 bps)**					
		A(4)		(34)	53682			A(4)		34	(52782)
			(21756)	3721	49997				21538	(3692)	(49120)
	A(2)	(0)					A(2)	(0)			
				(7531)	55578					7471	(54605)
			(15834)	(3753)	51887				15629	3722	(50935)
	(0)						(0)				
		A(5)		(12275)	57028			A(5)		38	(56014)
			(6722)	(8387)	53228				28813	(3819)	(52237)
A(1)		(5402)				A(1)		(4791)			
				(19038)	59182					7749	(58083)
			(1719)	(15126)	55378				22805	3867	(54300)
10487						(10612)					
		A(6)		(38)	58877			A(6)		38	(57814)
			(26908)	3991	54938				26648	(3958)	(53900)
	A(3)	(0)					A(3)	(0)			
				(8091)	61073					8023	(59923)
			(20644)	(4038)	57131				20399	4002	(56003)
	1304						(1326)				
		A(7)		(19905)	62735			A(7)		19598	(61534)
			(0)	(15728)	58670				(0)	15456	(57494)
		(10141)						10101			
				(26792)	65235					26410	(63932)
			4868	(22588)	61170				(4857)	22241	(59889)
Steepen up (40 bps)						**Flatten down (40 bps)**					
		A(4)		(51)	39716			A(4)		51	(39079)
			(9342)	1891	37884				9309	(1889)	(37236)
	A(2)	(0)					A(2)	(0)			
				(4182)	42201					4175	(41483)
			(7098)	(2219)	40374				7045	2213	(39641)
	(0)						(0)				
		A(5)		(3217)	43713			A(5)		57	(42957)
			(6722)	(1148)	41782				12467	(2011)	(41011)
A(1)		(1864)				A(1)		(787)			
				(7360)	46541					4452	(45687)
			(4682)	(5269)	44620				10142	2363	(43746)
5783						(5835)					
		A(6)		(57)	45005			A(6)		57	(44217)
			(11815)	2095	42993				11783	(2093)	(42190)
	A(3)	(0)					A(3)	0			
				(4627)	47878					4619	(46990)
			(9380)	(2454)	45877				9324	2447	(44969)
	(411)						400				
		A(7)		(9711)	49616			A(7)		9605	(48680)
			(0)	(7418)	47499				0	7314	(46543)
		(5340)						5355			
				(14009)	52893					13885	(51836)
			1896	(11694)	50793				(1918)	11571	(49711)

We demonstrate the hedging cost/benefit of the hypothetical insurance company reflecting the change of economic environment in the market with all the state contingent claims needed. In this paper, we further consider examples of the market current term structure parallel shift up/down (i.e. the parallel shift up 40 basis points case, changing I_A as 0.013737890; the parallel shift down 40 basis points case, changing I_A as 0.0062480852197), slop steepen/flatten (i.e. the steepen 40 basis points case, changing F_A as 0.019625268; the flatten 40 basis points case, changing F_A as 0.0003386181555) and the market current volatility structure parallel shift up/down (the parallel shift up 40 basis points case, changing $[\sigma]_A$ as [0.0540, 0.0540, 0.0540, 0.0540, 0.0540]; the parallel shift down 40 basis points case, changing $[\sigma]_A$ as [0.0460, 0.0460, 0.0460, 0.0460, 0.0460];), slop steepen/flatten (i.e. the steepen 40 basis points case, changing $[\sigma]_A$ as [0.0500, 0.0510, 0.0520, 0.0530, 0.0540]; the flatten 40 basis points, changing $[\sigma]_A$ as [0.0500, 0.0490, 0.0480, 0.0470, 0.0460]). The other parameters in the example of changing situations are the same as the original settings. Table 6 demonstrates the hedging strategy of the hypothetical insurance company while the market current term structure changes instantaneously. Positive numbers stand for long positions of the security, and numbers in the parentheses stand for short positions of the security. Similarly, while the volatility structure changes instantaneously, the hedging strategy of the hypothetical insurance company are shown in Table 7. Again, positive numbers stand for long positions of the security, and numbers in the parentheses stand for short positions of the security. Furthermore, the hypothetical insurance company could reach an optimal immunization with long/short hedging securities under the new economic environment. One may wonder how much value should the insurance company take to implement long and short the state contingent claims. We define hedging cost as the present value of long positions of state contingent claims minus the present value of short positions of state contingent claims. In other words, if hedging cost is positive, it stands for cash outflow, meanwhile hedging cost is negative, it stands for cash inflow, and hence, the hedging benefit. Further, we define reallocation value as the present value of long positions of state contingent claims plus the present value of short positions of state contingent claims. These results about hedging cost/benefit and reallocation value are shown in Table 8.

Table 7 Hedging Strategy While the Market Volatility Structure Changes

0	1	2	3	4	5	0	1	2	3	4	5
Parallel shift up (40 bps)						**Parallel shift down (40 bps)**					
		A(4)		7	(5412)			A(4)		(8)	5467
			23862	9595	(14979)				(23910)	(9701)	15136
	A(2)	(0)					A(2)	(0)			
				(23707)	(3894)					24010	3908
			46183	(14051)	(13513)				(46503)	14254	13625
	(0)						(0)				
		A(5)		(5731)	(2486)			A(5)		(3)	2484
			(6722)	4175	(12355)				17414	(9993)	12435
A(1)		33754				A(1)		(38751)			
				(29605)	(526)					24641	513
			15395	(19622)	(10450)				(5467)	14580	10515
(19651)						19662					
		A(6)		3	(570)			A(6)		(3)	565
			(7336)	10240	(10769)				7378	(10300)	10821
	A(3)	(0)					A(3)	(0)			
				(25029)	1442					25240	(1455)
			15896	(14714)	(8812)				(16060)	14871	8853
	27068						(27119)				
		A(7)		(24953)	3252			A(7)		25159	(3250)
			(0)	(14341)	(7297)				(0)	14517	7330
		14230						(14406)			
				(47328)	5817					47962	(5778)
			20113	(36627)	(4790)				(20576)	37239	4855
Steepen up (40 bps)						**Flatten down (40 bps)**					
		A(4)		9	(5687)			A(4)		(9)	5781
			25677	17074	(22734)				(25941)	(17322)	23071
	A(2)	(0)					A(2)	(0)			
				(34037)	(3527)					34744	3538
			57709	(16895)	(20626)				(58613)	17360	20876
	(0)						(0)				
		A(5)		(7129)	(2490)			A(5)		(4)	2491
			(6722)	10327	(19902)				20261	(17680)	20121
A(1)		36702				A(1)		(43343)			
				(41848)	298					35892	(323)
			25465	(24309)	(17163)				(13087)	18137	17354
(21297)						21499					
		A(6)		4	(2036)			A(6)		(4)	2036
			(48)	17778	(19766)				163	(17991)	19976
	A(3)	(0)					A(3)	(0)			
				(35904)	794					36417	(819)
			33291	(18046)	(16984)				(33674)	18352	17167
	22644						(22958)				
		A(7)		(24498)	2136			A(7)		24836	(2153)
			(0)	(6278)	(16003)				(0)	6435	16167
		20766						(20982)			
				(58112)	5749					59092	(5703)
			30402	(39801)	(12431)				(31102)	40607	12659

Table 8 Reallocation Assets and Cost/(Benefit) of the Hedging Strategy

Reallocation	Hedging cost/benefit	Reallocation	Hedging cost/benefit
Term parallel shift up (40 bps)		Term parallel shift down (40 bps)	
72973	31676	76667	(32322)
2.16%	0.94%	2.27%	-0.96%
Term steepen up (40 bps)		Term flatten down (40 bps)	
48130	27588	49442	(28150)
1.42%	0.82%	1.46%	-0.83%
Volatility parallel shift up (40 bps)		Volatility parallel shift down (40 bps)	
75157	(140)	76199	141
2.22%	0.00%	2.25%	0.00%
Volatility steepen up (40 bps)		Volatility flatten down (40 bps)	
87591	(106)	89567	106
2.59%	0.00%	2.65%	0.00%

Percent value is compared to the planned asset value.

One could see the largest hedging cost is 0.94% of planned asset value and the smallest hedging cost is 0.00% of planned asset value. Hence, When the economic environment changes slightly, the hedging cost is very small. However, some strategies are benefit, such as the largest benefit is 0.96% of planned asset value in case of the market current term structure parallel shifting down 40 basis points.

IV. CONCLUSION

This paper actually provides solid economic meanings of the “multi-period scenarios-based programming strategy”, which in essence is a profile of state contigent claims. In this paper, we have examined the immunization strategy for the surplus management of a hypothetical insurance company within a profile of state contingent claim approach. This hypothetical company could fulfill the liability schedule under pre-specified solvency ability. Most importantly, regardless the courses of interest rate level movements, the surplus value of the hypothetical insurance company always increases along with each path of asset return.

From the new point of view – a profile of state contingent claims, this immunization strategy actually decompose the “zero straight bond securities” as suggested by Tzeng, Wang, and Soo (2000) into the “state contingent claims” for asset allocation. If a surplus manager could long/short securities in the market with all the state contingent claims needed, he might obtain much more flexibility in the surplus management for an insurance company. Even, he can achieve path immunization for the surplus value in this market.

Furthermore, the hypothetical insurance company could reach an optimal immunization with long/short hedging securities under the new economic environment. Through the linear programming technique, the hypothetical insurance company could have its corresponding hedging strategy dealing with the change of the environment of the market current term/volatility structure. When the economic environment changes slightly, the hedging cost is very small. And, one can find that some strategies are benefits, not costs. We define two measures, such as hedging cost/benefit and reallocation value, to deal with it. Moreover, the numerical results show that the cost/benefit of hedging strategy via an optimal immunization strategy only reflects the change in value due to change in the market economy environment with all the state contingent claims needed.

Finally, with all the underlying state contingent claims, this paper illustrates the impact of the change of the market current term/volatility structure of asset/liability return on the surplus value, a way how to reallocate assets and a hedging strategy for this insurance company. And hence, the practitioners can refer our numerical implications for the surplus management.

APPENDIX

We take a four-period binomial BDT tree of asset return with one year per period as an example, to show reasonable number of state contingent claims. There are 16 kinds of paths in this BDT tree, denoted by the notations of ω_1, ω_2, … , ω_{16}, where each "scenario *i*" ω_i represents the *i*th path of BDT tree. We take "the scenario 11, ω_{11}" as an example to illustrate the concept of path. The "scenario 11, ω_{11}" represents a path of BDT tree as the up-down-up-down path. We can also denote the "scenario 11, ω_{11}" as the path of $r_0 - r_1^u - r_2^d - r_3^u - r_4^d$, where the subscript denotes as the index of period and the superscript denotes as the up state or the down state. In this notation, we can also see that total number of paths in a four-period binomial BDT tree is 16, say, $1\times 2\times 2\times 2\times 2$ = 16, due to the fact that the only two states can occur in each period of BDT tree.

However, this BDT tree generates 47 state contingent claims, that is 1+2+4+8+16+16 = 47 state contingent claims. In general, we can provide the general form of the number of state contingent claims with respect to the number of periods *n* in a *n*-period binomial interest rate model. The general form is $\sum_{k=0}^{n} 2^k + 2^n$. We take state 26 contingent claim as an example to illustrate the relationship between the state 26 and the scenario 11 in the four-period binomial BDT tree, and the payoff of state 26 contingent claim is shown as in Figure 1.

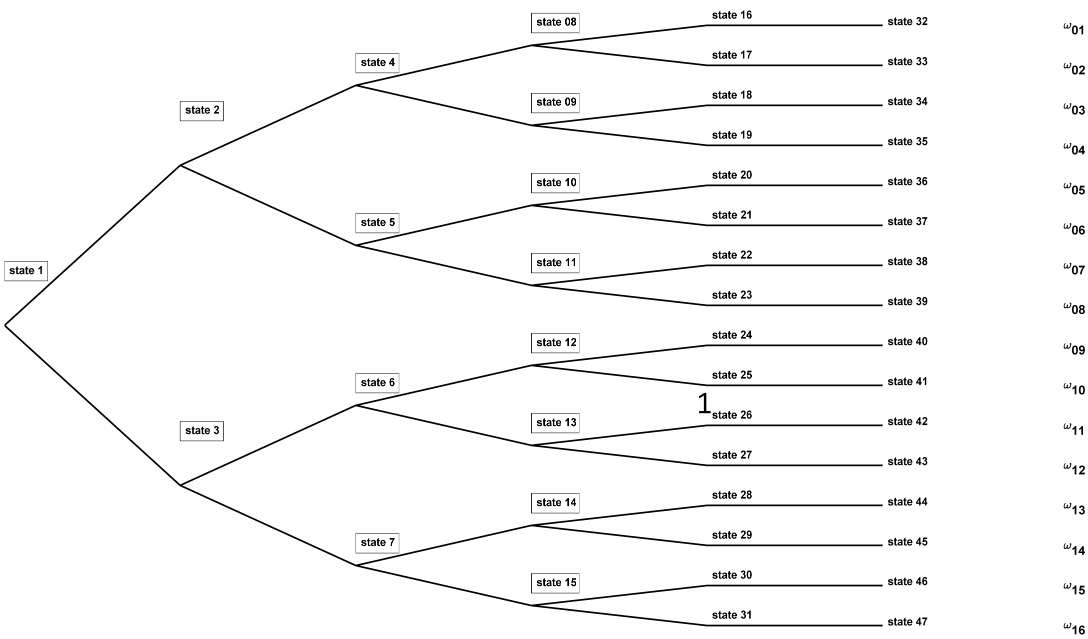

Figure 1 The payoff of the state 26 contingent claim

In the state 1 of Figure 1, we can denote it as the information set {ω_1, ω_2, … , ω_{16}}. The state 3 as {ω_9, ω_{10}, … , ω_{16}}, the state 6 as {ω_9, ω_{10}, … , ω_{12}}, the state 13 as {ω_{11}, ω_{12}}, the state 26 as {ω_{11}}. The path goes through the state 1 – the state 3 – the state 6 – state 13 – state 26 as to the corresponding path of "the scenario 11, ω_{11}", due to our definition of state as follows:

1. from the state 1 to the state 3 as the interest rate going up, also denoted by $r_0 - r_1^u$.
2. from the state 3 to the state 6 as the interest rate going down, also denoted by $r_1^u - r_2^d$.
3. from the state 6 to the state 13 as the interest rate going up, also denoted by $r_2^d - r_3^u$.
4. from the state 13 to the state 26 as the interest rate going down, also denoted by $r_3^u - r_4^d$.

Along with the definition of the state in this model, we can define the state 26 contingent claim whose payoff is illustrated by Figure 1. In this way, we can also define the state 1 contingent claim -- the state 31 contingent claim. In addition, due to the fact that one can determine the time 4 (the end of period 4) price of zero straight bond with maturity at time 5 given the known interest rate at the end of the last 4th period. We can extend to define the state at time 5, such as state 32 -- state 47 and its corresponding state contingent claims. Hence, this BDT tree generates 47 states and its corresponding state contingent claims.

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隨機模式下保險公司盈餘管理：狀態證券資產配置策略

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摘要

本文運用隨機利率模式所伴隨之一系列狀態證券，提供此隨機利率模式下最佳的資產配置策略，讓保險公司的盈餘管理者，依利率情境配置適當證券部位，使之能夠維持清償能力且滿足不同時期的現金支出。事實上，本文以一系列衍生性證券提供利率隨機模式下整體性免疫策略，亦即是，每一種情境，皆可對利率風險免疫。若考慮每一種情境免疫需求，則此資產配置模型可轉化為『多期的情境基礎規劃模式』。也就是說，本文經由狀態證券提供實際操作『情境基礎資產配置策略』的方法。另外，本文以衍生性證券為標的闡述『今日市場之即期利率期限結構』變動與『今日市場之即期利率波動期限結構』變動對保險公司盈餘價值的影響、保險公司如何重新配置資產、以及保險公司如何擬定避險策略。

關鍵字：盈餘管理，資產配置策略，狀態證券

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